§2.4) Taylor formula:

$f(z,w)$ – rational function with poles only on $z=0, w=0,$
$|z|=|w|$

$i_{z,w} f(z,w)$ – formal power series expansion in $|z|>|w|$

Let $a(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ be a formal distrib.

$$i_{z,w}\, a(z-w) \overset{\text{defn.}}{:=} \sum_{n\in\mathbb{Z}} a_n\, i_{z,w}\,(z-w)^n.$$

Kac says "$a(z-w)$ in region $|z|>|w|$"

Prop. 2.4

$a(z)$ – formal distribution. Then in $|z|>|w|$,

$$a(z+w) = \sum_{j=0}^{\infty} \partial^{(j)} a(z) w \qquad \text{where, } \partial^{(j)} = \frac{\partial^j}{j!}$$

Taylor formula but with formal distribution.

Proof:

$$i_{z,w}\, a(z+w) = \sum_n a_n\, i_{z,w}\,(z+w)^n = \sum_n a_n\, i_{z,w}\, z^n \left(1+\frac{w}{z}\right)$$

∵ we're working in $|z|>|w|$ region

$$= \sum_n a_n \sum_{j \geq 0} \binom{n}{j} z^{n-j} w^j$$

$$= \sum_{j \geq 0} \partial^{(j)} a(z) w^j$$

Then we make a change of variables $z \rightsquigarrow w$, $w \rightsquigarrow z-w$

$$a(z) = \sum_{j \geq 0} \partial^{(j)} a(w)(z-w)^j$$

(equality of formal distributions in w, $z-w$ in the region $|w| > |z-w|$)

Theorem-2.5(?)

$a(z)$ – formal distri

$N \geq 0$.

$$\left(\partial_w^N \delta(z-w)\right) a(z) = \left(\partial_w^N \delta(z-w)\right) \sum_{j=0}^{N} \partial^{(j)} a(w)(z-w)^j$$

Recall $\delta(z-w) = z^{-1} \sum_n \left(\frac{z}{w}\right)^n$ formally

Proof:

Recall that if $f(z)$ is a Laurent polynomial, $a(z,w)$ is a formal distribution,

$\mathrm{Res}_z\, a(z,w) f(z)$

$$\mathrm{Res}_z\left(\partial_z^N \delta(z-w)\, a(z) f(z)\right) = \mathrm{Res}_z\left(\partial_z^N \delta(z-w) \sum_j \partial^{(j)} a(w)\,(z-w)^j f(z)\right)$$

Identities frm. prev. talks:

1) $\partial_w \delta(z-w) = -\partial_z \delta(z-w)$

2) Also, $\mathrm{Res}_z(\partial\alpha\cdot\beta) = \mathrm{Res}_z(\alpha\cdot\partial\beta)$

$$\mathrm{Res}_z\left(\delta(z-w)\,\partial_z^N (a(z) f(z))\right) = \mathrm{Res}_z\left(\delta(z-w) \sum_j \partial^{(j)} a(w)\, \partial_z^N (z-w)^j f(z)\right)$$

3) $\mathrm{Res}_z\, \delta(z-w) f(z) = f(w)$ on LHS

4) $\partial^N(\alpha\cdot\beta) = \sum_k \binom{N}{k} \partial^k \alpha\, \partial^{N-k}\beta$ on RHS

$$\partial_w^N a(w) f(w) = \sum_j \frac{\partial^j}{j!} a(w) \binom{N}{j} \cdot j!\, \partial_w^{N-j} f(w)$$

$$= \sum_j \binom{N}{j} \partial^j a(w)\, \partial^{N-j} f(w)$$

§2.5) Current Algebras

$\mathfrak{h}$ - Lie superalgebra

$a(z), b(w)$: $\mathfrak{h}$-valued distributions

a, b are mutually local if one of the following equivalent conditions hold:

- $[a(z), b(w)] = \sum_{j=0}^{N-1} \partial_w^{(j)} \delta(z-w) c^j(w)$, c^j = some $\mathfrak{h}$-valued distri

- $a(z) b(w) = \sum_{j=0}^{N-1} i_{z,w} \frac{1}{(z-w)^{j+1}} c^j(w) + \underbrace{:a(z) b(w):}_{\left(\begin{array}{l} :a_n b_m: = \\ a_n b_m \text{ if } n<0 \\ b_m a_n \text{ if } n \geq 0 \end{array} \right)}$

Distri valued in univ. enveloping alg.

Physicists write

$$a(z) b(w) \sim \sum_j \frac{c^j(w)}{(z-w)^j}$$ – OPE, where

$c^j(w)$ are coefficients of OPE

$\boxed{\text{Equivalence of these is proven using computation from: } \partial^N \delta(z-w) = i_{z,w} \frac{1}{(z-w)^{N+1}} - i_{w,z} \frac{1}{(z-w)^{N+1}}}$

- $[a_{(m)}, b_{(n)}] = \sum_j \binom{m}{j} c^j_{m+n-j}(w) \, w^{m-j}$

• $[a_{(m)}, b(w)] = \sum_j \binom{m}{j} c^j(w) w^{m-j}$

I_0 - oscillator algebra

S - central extension of a ab alg. with basis

$\alpha(z) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1}$, $[\alpha(z), \alpha(w)] = \sum_n n z^{-n-1} w^{-n-1} K$

$= \partial_w \delta(z-w) K$

$\alpha(z)\alpha(w) \sim \frac{K}{(z-w)^2}$ — "free boson algebra"

$\mathfrak{g}$ - some Lie superalg.

$\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}]$ — Loop superalgebra

Assume $\mathfrak{g}$ has a super-symmetric invariant bilinear form:

$a|b = (-1)^{p(a)}(b|a)$ where p = parity

$\Downarrow$

$((a|b) = 0$ if $p(a) \neq p(b))$

$([a,b]|c) = (a|[b,c])$

$a_m = a \otimes t^n$, $a \in \mathfrak{g}$, $[a_m, b_n] = m(a|b)\delta_{m,n}K$ in $\hat{\mathfrak{g}}$

$\mathfrak{g}$

current alg.

$\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}K$

K is central in $\hat{\mathfrak{g}}$

If $\mathfrak{g}$ is a simple Lie alg, $(\,|\,)$ - Killing form,

$\hat{\mathfrak{g}}$ - Kac-Moody Lie alg. associated to $\mathfrak{g}$

$a(z) = \sum_n a_n z^{-n-1}$, $a \in \mathfrak{g}$ called currents.

Currents to be mutually local:

$$[a(z), b(w)] = \delta(z-w)[a,b](w) + \partial_w \delta(z-w)(a|b)K$$

or in OPE notation:

$$a(z)\,b(w) \sim \frac{[a,b](w)}{z-w} + \frac{(a|b)K}{(z-w)^2}$$

There is a generalization of affinization called superaffinization.

$\hat{\mathfrak{g}}_{super}$ is a central extension of the super loop algebra $\mathfrak{g}[t,t^{-1},\theta]$, $p(\theta)=1$

$$a_{n+\frac{1}{2}} = a\otimes t^n\theta, \quad \bar{a}(z) = \sum_{n\in\mathbb{Z}} a_{n+\frac{1}{2}} z^{-n-1} \text{ – supercurrent.}$$

$\bar{a}(z)$ are mutually local wrt both currents & supercurrents.

$$a(z)\overline{b(w)} \sim \frac{\overline{[a,b]}(w)}{z-w}, \qquad \overline{a(z)}\,\overline{b(w)} = \frac{(b|a)K}{z-w}$$

A – super vector space with super-skew symm. form:

$$(a|b) = -(-1)^{p(a)}(b|a)$$

Clifford affinization is the superalgebra $C_A = A[t,t^{-1}]\oplus\mathbb{C}K$ with commutation relations

$$[\varphi_m, \psi_n] = (\varphi,\psi)\delta_{m,-n}K, \quad m,n\in\tfrac{1}{2}+\mathbb{Z}$$

$$\varphi,\psi\in A$$

$$\alpha_m = \alpha\otimes t^{m-1/2}$$

$\varphi(z) = \sum \varphi_{n+\frac{1}{2}} z^{-n-1}$ are mutually local.

$$\varphi(z)\psi(w) \sim \frac{(\varphi,\psi)K}{z-w}$$

<u>Example-1</u>:

$$A = \mathrm{span}_{\mathbb{C}}\langle\varphi\rangle, \quad (\varphi|\varphi) = 1$$
$$K = 1$$

In C_A, $\varphi_m\varphi_n + \varphi_n\varphi_m = \delta_{m,-n}$, $\varphi(z)\varphi(w) \sim \dfrac{1}{z-w}$

neutral free fermion

<u>Example-2</u>:

$$A = \mathrm{span}\langle\varphi^+, \varphi^-\rangle, \quad (\varphi^+|\varphi^-) = 1, \quad (\varphi^\pm, \varphi^\pm) = 0,$$

$$\varphi^\pm_m \varphi^\mp_n + \varphi^\mp_n \varphi^\pm_m = \delta_{m,-n}, \quad \varphi^\pm_m \varphi^\pm_n + \varphi^\pm \varphi^\pm_m = 0$$

$$\varphi^\pm(z)\,\varphi^\mp(w) \sim \frac{1}{z-w}$$

$$\varphi^\pm(z)\varphi^\mp(w) = 0$$

$\varphi^\pm(z)$ – charged free fermions.

§ 2.6) Conformal weights & Virasoro alg.

U - some assoc. alg

H - derivation of U ("Hamiltonian")

$a(z,w,\dots)$ is called an eigendistribution with conformal weight Δ if

$$(H - \Delta - z\partial_z - w\partial_w - \cdots)a = 0$$

<u>Note</u>: if $a = a(z) = \sum a_n z^n$,

$$(H - \Delta - n)\, a_n = 0$$

<u>Proposition</u>:

$a, b, \dots, w$ weights Δ, Δ'

a) $\partial_z a$ is an eigendistribution of conformal weight $\Delta + 1$

b) $:a(z) b(w):$ " " " " " $\Delta + \Delta'$

c) n^{th} op. coeff. in $[a(z), b(w)]$ has weight $\Delta + \Delta' - n - 1$

d) f - homog. fn. of degree j, then fa is a "

" " " with wt. $\Delta - j$

Corollary:

a, b mutually local of wt. $\Delta, \Delta' \Rightarrow$

$a(z)b(w) \sim \sum \frac{c^j(w)}{(z-w)^{j+1}}$ — all summands of deg. $\Delta + \Delta'$

Convention:

a is of weight Δ, $a = \sum_{n \in -\Delta + \mathbb{Z}} a_n z^{-n-\Delta}$

$$[a_m, b_n] = \sum_{j=0}^{N-1} \binom{m+\Delta-1}{j} c^j_{m+n}$$

$$\boxed{[a_m, b(z)] = \sum_{j=0}^{N-1} \binom{m+\Delta-1}{j} c^j(z)\, z^{m+\Delta-j-1}}$$

Example:

For $\mathbb{A}$, $H = -t\partial_t$ $\left(H = -t\partial_t - \frac{1}{2}\theta\partial\theta\right)$

currents have wt. 1, supercurrents $\frac{1}{2}$.

It commonly happens that conformal weights are $\frac{1}{2}\mathbb{Z}^{+}$, with weight 0, achieved by central element which leads to:

wt(a), wt(b)	$a(z)b(w) \sim$
$\frac{1}{2}, \frac{1}{2}$	$\frac{c}{z-w} + \frac{c}{(z-w)^2}$
$1, 1$	$\frac{c(w)}{z-w} + \frac{c}{(z-w)^2}$
$\frac{1}{2}, 1$	$\frac{c_{1/2}(w)}{z-w}$

We look for an even self-local distribution of conformal weight 2:

$$L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

This restriction implies,

$$L(z)L(w) = \frac{c/2}{(z-w)^4} + \frac{a(w)}{(z-w)^3} + \frac{2b(w)}{(z-w)^2} + \frac{c(w)}{z-w}$$

<u>Thm:</u> If we have such L, then:

a) $a(w)=0$, $c(w)=\partial b(w)$

b) If in addition, $[C, L(z)]=0$,

$$[L_{-1}, L(z)] = \partial L(z)$$

$$[L_0, L(z)] = (z\partial+2)L(z)$$

Then, coeff. L_n satisfy Virasoro relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C.$$